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Quantum group covariant (anti)symmetrizers, ε-tensors, vielbein, Hodge map and Laplacian

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Abstract

$GL_q(N)$- and $SO_q(N)$-covariant deformations of the completely symmetric/antisymmetric projectors with an arbitrary number of indices are explicitly constructed as polynomials in the braid matrices. The precise relation between the completely antisymmetric projectors and the completely antisymmetric tensor is determined. Adopting the $GL_q(N)$- and $SO_q(N)$-covariant differential calculi on the corresponding quantum group covariant noncommutative spaces $\mathbb{C}_q^N$, $\mathbb{R}_q^N$, we introduce a generalized notion of vielbein basis (or 'frame'), based on differential-operator-valued 1-forms. We then give a thorough definition of a $SO_q(N)$-covariant $\mathbb{R}_q^N$-bilinear Hodge map acting on the bimodule of differential forms on $\mathbb{R}_q^N$, introduce the exterior coderivative and show that the Laplacian acts on differential forms exactly as in the undeformed case, namely it acts on each component as it does on functions.

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1. Introduction and preliminaries

The noncommutative geometry programme [4, 7] and the related programme of generalizing the concept of symmetries through quantum groups [6, 8, 25] and quantum group covariant noncommutative spaces (shortly quantum spaces) [8, 19] has found a widespread interest in the mathematical and theoretical physics community over the past two decades for its potential applications in both fundamental and applied physics. In order to make either programme powerful on a specific model it is important to reproduce as many of the tools available in the corresponding undeformed (commutative) geometry model (if any) as possible. The

scope of the present work is to revisit and/or solve a number of related technical issues, left (partially or totally) untreated or unsolved in the literature, regarding the quantum groups $H = U_q gl(N), U_q so(N)$ of the classical series [6, 8], the noncommutative spaces $\mathbb{C}_q^N, \mathbb{R}_q^N$ [8, 19] on which they act, and the quantum group covariant differential calculi [1, 22, 24] on the latter.

As known, the braid matrix $\hat{R}$ of H [8] is a $N^2 \times N^2$ matrix, H-covariant deformation of the permutation matrix P. H-covariant (anti)symmetrizers $\mathcal{P}^{\pm}$ of 2-tensors arise from the projector decomposition of $\hat{R}$, or equivalently can be expressed as (first or second degree) polynomials in $\hat{R}$. In analogy with the undeformed case, H-covariant (anti)symmetrizers $\mathcal{P}^{\pm,l}$ of l-tensors ($l \geqslant 2$) are expected (see, e.g., [9, 11]) to be polynomials in $\hat{R}_{12}, \ldots, \hat{R}_{(l-1)l}$, the matrices obtained as tensor products of $\hat{R}$ with $l-2$ copies of the $N \times N$ unit matrix. In section 2 we find a very compact and manageable recursive relation, through which these polynomials are determined. For $H = U_q so(N)$ this is in agreement with the much longer recursive relation found in [15][1]. In section 3 we recall or prove properties of the H-covariant ε-tensor [8, 10, 16–18] and determine precisely its relation with the antisymmetric projectors $\mathcal{P}^{-,l}$. In section 4 we introduce an open-minded generalization of the notion [5] of vielbein (or 'frame') basis of 1-forms on $\mathbb{C}_q^N, \mathbb{R}_q^N$; we modify the approach adopted for $\mathbb{R}_q^N$ in [2], in that we allow the matrix transforming the basis $\{\mathrm{d}x^i\}$ into the vielbein to have as entries differential operators, rather than functions[2]. In section 5 we introduce a thorough and consistent definition of a bilinear, $U_q so(N)$-covariant Hodge map, exterior coderivative and Laplacian acting on differential forms on $\mathbb{R}_q^N$.

The projector decomposition of the H-covariant braid matrix $\hat{R}$ reads

$$\hat{R} = q\mathcal{P}^+ - q^{-1}\mathcal{P}^-, \qquad \text{if } H = U_q gl(N), \tag{1.1a}$$

$$\hat{R} = q\mathcal{P}^+ - q^{-1}\mathcal{P}^- + q^{1-N}\mathcal{P}^t, \qquad \text{if } H = U_q so(N). \tag{1.1b}$$

$\mathcal{P}^-$ is the corresponding deformation of the antisymmetric projector. In (1.1a) the matrix $\mathcal{P}^+$ is the $U_q gl(N)$-covariant deformation of the symmetric projector, in (1.1b) it is the $U_q so(N)$-covariant deformation of the symmetric trace-free projector, while $\mathcal{P}^t$ is the trace projector. Thus they satisfy the equations

$$\mathcal{P}^\alpha \mathcal{P}^\beta = \mathcal{P}^\alpha \delta^{\alpha\beta}, \qquad \sum_\alpha \mathcal{P}^\alpha = \mathbf{1}_{N^2}, \tag{1.2}$$

where $(\mathbf{1}_{N^2})^{ij}_{hk} = \delta^i_h \delta^j_k$, $\alpha, \beta = -, +$ in the $H = U_q gl(N)$ case and $\alpha, \beta = -, +, t$ in the $H = U_q so(N)$ case. $\hat{R}$ is a symmetric matrix, and therefore also the projectors are

$$\hat{R}^T = \hat{R}, \qquad \mathcal{P}^{\alpha\,T} = \mathcal{P}^\alpha. \tag{1.3}$$

The braid matrix fulfils the equation

$$f(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12} f(\hat{R}_{23}) \tag{1.4}$$

for any rational function $f(t)$ in one variable such that the spectrum of $f(\hat{R})$ has no poles, in particular for $f(\hat{R}) = \hat{R}, \hat{R}^{-1}, \mathcal{P}^\alpha$. Here we have used the conventional matrix-tensor notation $\hat{R}_{12} = \hat{R} \otimes \mathbf{1}_N$, $\hat{R}_{23} = \mathbf{1}_N \otimes \hat{R}$, where $\mathbf{1}_N$ denotes the $N \times N$ unit matrix.

In the $H = U_q so(N)$ case the $\mathcal{P}^t$ projects on a one-dimensional sub-space and can be written in the form

$$\mathcal{P}^{t\,ij}_{kl} = \frac{1}{Q_N} g^{ij} g_{kl}, \tag{1.5}$$

[1] We thank the authors of [15] for calling our attention to their paper, which we did not know, after the appearance of the first version of the present work on the electronic arXive.

[2] As a by-product some unpleasant aspects of the vielbein of [2] disappear. Incidentally, this change of attitude should allow us to introduce a frame basis also for other quantum spaces, notably q-Minkowski.

where the $N \times N$ matrix g_{ij} is a $U_q so(N)$-isotropic tensor, deformation of the ordinary Euclidean metric, which will be given in (3.10), and [1]

$$Q_N \equiv g^{lm} g_{lm} = \frac{(1+q^{2-N})(q^N-1)}{q^2-1} = \left(q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1}\right) \left[\frac{N}{2}\right]_q .$$

Here and in the following we use the 'q-deformed numbers'

$$[y]_q := \frac{q^y - q^{-y}}{q - q^{-1}}, \qquad y_{q^{\pm 2}} := \frac{q^{\pm 2y} - 1}{q^{\pm 2} - 1} = q^{\pm(l-1)}[y]_q . \tag{1.6}$$

The metric and the braid matrix satisfy the relations [8]

$$g_{il} \hat{R}^{\pm 1\, lh}_{\ \ jk} = \hat{R}^{\mp 1\, hl}_{\ \ ij} g_{lk}, \qquad g^{il} \hat{R}^{\pm 1\, jk}_{\ \ lh} = \hat{R}^{\mp 1\, ij}_{\ \ hl} g^{lk}. \tag{1.7}$$

Indices will be lowered and raised using g_{ij} and its inverse g^{ij}, e.g.

$$\partial^i := g^{ij}\partial_j \qquad x_i := g_{ij} x^j .$$

By taking powers of either decomposition (1.1) one can express the projectors as polynomials in $\hat{R}$. One finds

$$\mathcal{P}^{\pm} = \frac{q^{\mp 1}\mathbf{1}_{N^2} \pm \hat{R}}{q + q^{-1}} \qquad \text{if } H = U_q gl(N), \tag{1.8}$$

$$\mathcal{P}^{\pm} = \frac{q^{\mp 1}\mathbf{1}_{N^2} \pm \hat{R} - (q^{\mp 1} \pm q^{1-N})\mathcal{P}^t}{q + q^{-1}} \qquad \text{if } H = U_q so(N). \tag{1.9}$$

The deformed algebras of functions on the two quantum spaces are called 'algebra of functions on the quantum hyperplane $\mathbb{C}^N_q$' and 'algebra of functions on the quantum Euclidean space $\mathbb{R}^N_q$' respectively ($h := \ln q$ plays the role of deformation parameter); we shall denote either one by F. F is essentially the unital associative algebra over $\mathbb{C}[[h]]$ generated by N elements x^i (the Cartesian 'coordinates') modulo the relations (1.10) given below. The corresponding H-covariant differential calculi [1, 24] are defined introducing the invariant exterior derivative d, satisfying nilpotency and the Leibniz rule $\mathrm{d}(fg) = \mathrm{d}f g + f\mathrm{d}g$, and imposing the covariant commutation relations (1.11) between the x^i and the differentials $\xi^i := \mathrm{d}x^i$. Partial derivatives are introduced through the decomposition $\mathrm{d} =: \xi^i \partial_i$. All the other commutation relations are derived by consistency. The complete list is given by

$$\mathcal{P}^{-\,ij}_{\ \ hk} x^h x^k = 0, \tag{1.10}$$

$$x^h \xi^i = q \hat{R}^{hi}_{\ \ jk} \xi^j x^k, \tag{1.11}$$

$$(\mathbf{1}_{N^2} - \mathcal{P}^-)^{ij}_{hk} \xi^h \xi^k = 0, \tag{1.12}$$

$$\mathcal{P}^{-\,ij}_{\ \ hk} \partial_j \partial_i = 0, \tag{1.13}$$

$$\partial_i x^j = \delta^j_i + q \hat{R}^{jh}_{ik} x^k \partial_h, \tag{1.14}$$

$$\partial_h \xi^i = q^{-1} \hat{R}^{-1\, ik}_{\ \ hj} \xi^j \partial_k . \tag{1.15}$$

We shall call $\mathcal{DC}^*$ (differential calculus algebra on $\mathbb{R}^N_q$) the unital associative algebra over $\mathbb{C}[[h]]$ generated by x^i, ξ^i, ∂_i modulo these relations. We shall denote by $\bigwedge^*$ (exterior algebra, or algebra of exterior forms) the graded unital subalgebra generated by the ξ^i alone, with grading $\natural \equiv$ the degree in ξ^i, and by $\bigwedge^p$ (vector space of exterior p-forms) the component with grading $\natural = p$, $p = 0, 1, 2, \ldots$. Each $\bigwedge^p$ carries an irreducible representation of H, and its dimension is the binomial coefficient $\binom{N}{p}$ [8, 10], exactly as in the $q = 1$ (i.e. undeformed)

case; in particular there are no forms with $p > N$, and $\dim(\bigwedge^N) = \binom{N}{N} = 1$, therefore $\bigwedge^N$ carries the singlet representation of H.

We shall endow $\mathcal{DC}^*$ with the same grading $\natural$, and call $\mathcal{DC}^p$ its component with grading $\natural = p$. The elements of $\mathcal{DC}^p$ can be considered differential-operator-valued p-forms.

We shall denote by Ω^* (algebra of differential forms) the graded unital subalgebra generated by the ξ^i, x^i, with grading $\natural$, and by Ω^p (space of differential p-forms) its component with grading p; by definition $\Omega^0 = F$ itself. Clearly both Ω^* and Ω^p are F-bimodules.

We shall denote by $\mathcal{H}$ (Heisenberg algebra) the unital subalgebra generated by the x^i, ∂_i. Note that by definition $\mathcal{DC}^0 = \mathcal{H}$, and that both $\mathcal{DC}^*$ and $\mathcal{DC}^p$ are $\mathcal{H}$-bimodules. Finally, we shall call F' the unital associative algebra generated by the ∂_i alone.

In the $H = U_q so(N)$-covariant case the elements

$$r^2 := x \cdot x = g_{kl} x^k x^l, \qquad \Box := \partial \cdot \partial = g^{kl} \partial_l \partial_k = g_{kl} \partial^k \partial^l$$

are $U_q so(N)$-invariant and respectively generate the centres of F, F'.

The H-covariance of the differential calculus implies that $\mathcal{DC}^*$ is a (right, in our conventions) H-module algebra. All the information on the algebras $\mathcal{DC}^*$, H and the right action of the Hopf algebra H on $\mathcal{DC}^*$ can be encoded in the cross-product algebra $\mathcal{DC}^* \rtimes H$. We recall that this is $H \otimes \mathcal{DC}^*$ as a vector space, and so we denote as usual $g \otimes a$ simply by ga; that $H\mathbf{1}_{\mathcal{DC}^*}$, $\mathbf{1}_H \mathcal{DC}^*$ are subalgebras isomorphic to H, $\mathcal{DC}^*$, and so we omit to write either unit $\mathbf{1}_{\mathcal{DC}^*}$, $\mathbf{1}_H$ whenever multiplied by non-unit elements; that for any $a \in \mathcal{DC}^*$, $g \in H$ the product fulfils

$$ag = g_{(1)}(a \triangleleft g_{(2)}). \tag{1.16}$$

Here $\Delta(g) = g_{(1)} \otimes g_{(2)}$ denotes the coproduct of g in Sweedler notation. $\mathcal{DC}^* \rtimes H$ is a H-module algebra itself, if we extend $\triangleleft$ on H as the adjoint action, namely as

$$h \triangleleft g = S g_{(1)} h g_{(2)}.$$

In view of (1.16), this formula will correctly reproduce the action also on the elements of $\mathcal{DC}^*$, and therefore on *any* element $a \in \mathcal{DC}^* \rtimes H$. The elements σ^i, with $\sigma^i = x^i, \xi^i, \partial^i$, transform with the N-dimensional representation ρ of $U_q sl(N)$ or $U_q so(N)$ respectively:

$$\sigma^i \triangleleft g = \rho^i_j(g) \sigma^j. \tag{1.17}$$

The above scheme applies also to the Hopf algebra $H = \widetilde{U_q so(N)}$, which is the central extension of $H = U_q so(N)$ obtained by adding a central and primitive element η generating dilatations of elements of $\mathcal{H}$,

$$x^i \eta = (\eta + 1) x^i, \qquad \xi^i \eta = (\eta + 1) \xi^i, \qquad \partial^i \eta = (\eta - 1) \partial^i. \tag{1.18}$$

We shall call η also the generator of dilatations of $U_q gl(N)$.

One can introduce an alternative H-covariant differential calculus replacing $q\hat{R}$ by $(q\hat{R})^{-1}$ in the defining relations (1.11)–(1.15). The corresponding objects $\hat{\xi}^i$, $\hat{\partial}_i$ can be realized as suitable 'functions' of x^j, ξ^j, ∂_j [20].

2. Completely (anti)symmetric projectors

The projectors $\mathcal{P}^{\pm,l} \equiv \left\| \mathcal{P}^{\pm,l}{}^{i_1 \dots i_l}_{j_1 \dots j_l} \right\|$ project the tensor product of l copies of the N-dimensional representation of H to the l-fold completely symmetric/antisymmetric irreducible representation $V_N^{\pm,l}$ of H therein contained. They are uniquely characterized by the following properties,

$$\mathcal{P}^{\pm,l} \mathcal{P}^{\pi}_{m(m+1)} = \delta^{\pm}_{\pi} \mathcal{P}^{\pm,l} = \mathcal{P}^{\pi}_{m(m+1)} \mathcal{P}^{\pm,l}, \tag{2.1}$$

$$(\mathcal{P}^{\pm,l})^2 = \mathcal{P}^{\pm,l}, \tag{2.2}$$

$$\mathrm{tr}_{1\ldots l}(\mathcal{P}^{\pm,l}) = \dim\left(V_N^{\pm,l}\right), \tag{2.3}$$

where $\pi = -, +$ in the $U_q sl(N)$- and $\pi = -, +, t$ in the $U_q so(N)$-covariant case respectively, $m = 1, \ldots, l-1$ and by $\mathcal{P}^{\pi}_{m(m+1)}$ we have denoted the matrix acting as $\mathcal{P}^{\pi}$ on the mth, $(m+1)$th indices and as the identity on the remaining ones. Equation (2.3) guarantees that $\mathcal{P}^{\pm,l}$ act as the identity (and not as proper projectors) on $V_N^{\pm,l}$; for both $H = U_q gl(N), U_q so(N)$ $\dim\left(V_N^{-,l}\right) = \binom{N}{l}$, whereas for $H = U_q gl(N)$ $\dim\left(V_N^{+,l}\right) = \binom{N-1+l}{N-1}$, for $H = U_q so(N)$ $\dim\left(V_N^{+,l}\right) = \binom{N-1+l}{N-1} - \binom{N-3+l}{N-1}$. In the appendix we prove:

Proposition 1. *The projectors $\mathcal{P}^{\pm,l+1}$ can be expressed as polynomials in $\hat{R}_{12}, \ldots, \hat{R}_{(l-1)l}$ through either recursive relation*

$$\mathcal{P}^{\pm,l+1} = \mathcal{P}^{\pm,l}_{12\ldots l} M^{\pm,l+1}_{l(l+1)} \mathcal{P}^{\pm,l}_{12\ldots l}, \tag{2.4}$$

$$= \mathcal{P}^{\pm,l}_{2\ldots(l+1)} M^{\pm,l+1}_{12} \mathcal{P}^{\pm,l}_{2\ldots(l+1)}, \tag{2.5}$$

where $\mathcal{P}^{\pm,l}_{1\ldots l} \equiv \mathcal{P}^{\pm,l} \otimes \mathbf{1}_N$, $\mathcal{P}^{\pm,l}_{2\ldots(l+1)} \equiv \mathbf{1}_N \otimes \mathcal{P}^{\pm,l}$, $M^{\pm,l+1}_{l(l+1)} = \mathbf{1}_N^{\otimes^{l-1}} \otimes M^{\pm,l+1}$ etc, and

$$\begin{aligned} M^{\pm,l+1} &= \frac{1}{[l+1]_q}[q^{\mp l}\mathbf{1}_{N^2} \pm [l]_q \hat{R}] && \text{if } H = U_q sl(N), \\ M^{\pm,l+1} &= \frac{1}{[l+1]_q}\left[q^{\mp l}\mathbf{1}_{N^2} \pm [l]_q \hat{R} + \frac{Q_N(q^{\pm 2}-1)[l]_q}{q^{\pm 1} \mp q^{N-1\pm 2l}}\mathcal{P}^t\right] && \text{if } H = U_q so(N). \end{aligned} \tag{2.6}$$

As a consequence, they are symmetric, $(\mathcal{P}^{\pm,l})^T = \mathcal{P}^{\pm,l}$, and if $H = U_q so(N)$

$$\mathcal{P}^{\pm,l}{}^{i_1\ldots i_l}_{j_1\ldots j_l} g^{j_1k_1} \ldots g^{j_lk_l} = g_{i_1j_1} \ldots g_{i_lj_l} \mathcal{P}^{\pm,l}{}^{j_l\ldots j_1}_{k_l\ldots k_1}, \tag{2.7}$$

and the same with the matrix g^{ij} replaced by its transpose g^{ji}.

In [9] we explicitly determined as examples just $\mathcal{P}^{\pm,3}$ for $H = U_q so(N)$. In [15] longer recursive relations for $\mathcal{P}^{\pm,l}$ with arbitrary l in the case $H = U_q so(N)$ were given; the ansatz adopted there was of the type $\mathcal{P}^{\pm,l+1} = B^{\pm,l+1}\mathcal{P}^{\pm,l}_{12\ldots l}$. The unknown $N^{l+1} \times N^{l+1}$ matrices $B^{\pm,l+1}$ were explicitly determined to be rather long polynomials in $\hat{R}_{12}, \ldots, \hat{R}_{(l-1)l}$. To go from our formula to theirs one just needs to set $B^{\pm,l+1} = \mathcal{P}^{\pm,l}_{12\ldots l} M^{\pm,l+1}_{l(l+1)}$; to go from their formula to ours one has to multiply both sides by $\mathcal{P}^{\pm,l}_{12\ldots l}$ from the left and do a straightforward calculation using (2.1), (1.1*b*).

Remark. One can easily check that in the $H = U_q gl(N)$ case the deformed (anti)symmetric projectors $\mathcal{P}^{\pm,l}$ can be obtained from the polynomials giving the undeformed (anti)symmetric projectors in terms of the permutators $P_{m(m+1)}$ by replacing the latter respectively with $\pm q^{\pm 1}\hat{R}_{m(m+1)}$, and readjusting the normalizations.

3. Properties of the *H*-covariant ε-tensors

In our convention indices $i, j, \ldots$ take the following values:

$$i = 1, 2, \ldots, N \qquad \text{if } H = U_q gl(N), \tag{3.1}$$

$$i = -n, \ldots, -1, 0, 1, \ldots, n \qquad \text{if } H = U_q so(2n+1), \tag{3.2}$$

$$i = -n, \ldots, -1, 1, \ldots, n \qquad \text{if } H = U_q so(2n). \tag{3.3}$$

Then the commutation relations (1.12) explicitly amount to

$$q\xi^i\xi^j + \xi^j\xi^i = 0 \qquad i < j \neq -i, \tag{3.4a}$$

$$\xi^i\xi^i = 0 \qquad i \neq 0, \tag{3.4b}$$

$$\xi^l\xi^{-l} + \xi^{-l}\xi^l = (q - q^{-1}) \sum_{i>l} q^{l+1-i}\xi^{-i}\xi^i \qquad l \geqslant 1, \tag{3.4c}$$

$$\xi^0\xi^0 = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) \sum_{i>0} q^{1-i}\xi^{-i}\xi^i. \tag{3.4d}$$

Of course (3.4*d*) applies only to the cases $H = U_q so(2n+1)$, and (3.4*c*) applies only to the cases $H = U_q so(N)$. The latter relations are equivalent to the equations (21) given in [10], whence they can be obtained by an easy rearrangement of terms.

As already said, as a consequence of (3.4) $\dim(\bigwedge^N) = 1$. Setting e.g.

$$\begin{aligned} &\gamma_N\, \mathrm{d}^N x := \xi^1\xi^2 \ldots \xi^N && \text{if } H = U_q gl(N) \\ &\gamma_N\, \mathrm{d}^N x := \xi^{-n}\xi^{1-n} \ldots \xi^n && \text{if } H = U_q so(N) \end{aligned} \tag{3.5}$$

one can introduce the matrix elements of the H-covariant ε- (or completely antisymmetric) tensor up to the normalization constant γ_N by the relation

$$\xi^{i_1}\xi^{i_2} \ldots \xi^{i_N} = \mathrm{d}^N x\, \varepsilon^{i_1 i_2 \ldots i_N}. \tag{3.6}$$

The ε-tensors enter the definitions of the 'q-determinants' [8, 10], special central elements in the Hopf algebras H' dual to H, namely the algebras of functions on the quantum groups. In the appendix we prove the following proposition, which states a similar property for the q-determinant of the matrices $\mathcal{L}^{\pm}$ having as matrix elements the so-called FRT (Faddeev–Reshetikin–Takhtadjan) generators [8] of $U_q sl(N)$, $U_q so(N)$:

$$\mathcal{L}^{+,a}_{\ l} := \mathcal{R}^{(1)}\rho^a_l(\mathcal{R}^{(2)}), \qquad \mathcal{L}^{-,a}_{\ l} := \rho^a_l(\mathcal{R}^{-1(1)})\mathcal{R}^{-1(2)}; \tag{3.7}$$

here $\mathcal{R}$ denotes the quasitriangular structure.

Proposition 2.

$$\begin{aligned} &\mathcal{L}^{+,j_N}_{\ i_N} \ldots \mathcal{L}^{+,j_1}_{\ i_1} \varepsilon^{i_1 \ldots i_N} = \varepsilon^{j_1 \ldots j_N}, \\ &\mathcal{L}^{-,j_N}_{\ i_N} \ldots \mathcal{L}^{-,j_1}_{\ i_1} \varepsilon^{i_1 \ldots i_N} = \varepsilon^{j_1 \ldots j_N}. \end{aligned} \tag{3.8}$$

In particular

$$\begin{aligned} &\det{}_q\, \mathcal{L}^{\pm} := \mathcal{L}^{\pm,N}_{\ i_N} \ldots \mathcal{L}^{\pm,1}_{\ i_1} \varepsilon^{i_1 \ldots i_N} = \gamma_N && \text{if } H = U_q sl(N), \\ &\det{}_q\, \mathcal{L}^{\pm} := \mathcal{L}^{\pm,n}_{\ i_N} \ldots \mathcal{L}^{\pm,-n}_{\ i_1} \varepsilon^{i_1 \ldots i_N} = \gamma_N && \text{if } H = U_q so(N). \end{aligned} \tag{3.9}$$

The $U_q so(N)$-covariant matric matrix introduced in (1.5) coincides with its inverse and is given by

$$g_{ij} = g^{ij} = q^{\rho_j}\delta_{-i,j}, \tag{3.10}$$

where $(2\rho_j) := (N-2, N-4, \ldots, 1, 0, -1, \ldots, 2-N)$ for N odd, $(2\rho_j) := (N-2, N-4, \ldots, 0, 0, \ldots, 2-N)$ for N even. Introducing the matrix U by

$$\begin{aligned} &U^i_j = \delta^i_j q^{2i-N-1} && \text{if } H = U_q gl(N) \\ &U^i_j := g^{ik}g_{jk} = \delta^i_j q^{-2\rho_i} && \text{if } H = U_q so(N) \end{aligned} \tag{3.11}$$

(note that $\det U = 1$), we can also recall the q-cyclic property [23][3]

$$\epsilon^{i_1 \ldots i_N} = (-1)^{N-1} U^{i_i}_{j_1} \epsilon^{i_2 \ldots i_N j_1}. \tag{3.12}$$

[3] The proof given in [23] applies also to the $H = U_q gl(N)$ case.

Let $I := (i_1, \dots, i_N)$, and if I is a permutation of $J := (1, \dots, N)$ denote by $l(I)$ its 'length', namely its number of inversions. The $U_q gl(N)$-covariant deformation of the ε-tensor [8, 16, 17] admits the following compact expression, which closely resembles the undeformed counterpart

$$\varepsilon^{i_1 i_2 \dots i_N} = \begin{cases} (-q)^{l(I)} & \text{if } I \text{ is a permutation of } J, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

For the $U_q so(N)$-covariant one [10] so far no such compact expression has been found. In [10, 18] several properties for general N and the explicit expression for ε in the cases $N = 3, 4$ have been determined; we rewrite them here: for $N = 3$, with normalization $\gamma_3 = q^{-1}$

$$\begin{array}{llll} \varepsilon^{-101} = q^{-1} & \varepsilon^{-110} = -1 & \varepsilon^{0-11} = -1 & \varepsilon^{01-1} = 1 \\ \varepsilon^{10-1} = -q & \varepsilon^{1-10} = 1 & \varepsilon^{000} = 1/\sqrt{q} - \sqrt{q} & \varepsilon^{ijk} = 0 \text{ otherwise,} \end{array}$$

and for $N = 4$, with normalization $\gamma_4 = q^{-2}$

$$\begin{array}{llll} \varepsilon^{-2-112} = q^{-2} & \varepsilon^{-21-12} = -q^{-2} & \varepsilon^{-2-121} = -q^{-1} & \varepsilon^{-212-1} = q^{-1} \\ \varepsilon^{-22-11} = 1 & \varepsilon^{-221-1} = -1 & \varepsilon^{-1-212} = -q^{-1} & \varepsilon^{-11-22} = 1 \\ \varepsilon^{-1-221} = 1 & \varepsilon^{-12-21} = -1 & \varepsilon^{-121-2} = q & \varepsilon^{-112-2} = -1 \\ \varepsilon^{1-1-22} = -1 & \varepsilon^{1-2-12} = q^{-1} & \varepsilon^{1-12-2} = q & \varepsilon^{12-1-2} = -q \\ \varepsilon^{12-2-1} = 1 & \varepsilon^{1-22-1} = -1 & \varepsilon^{2-2-11} = -1 & \varepsilon^{2-1-21} = q \\ \varepsilon^{21-2-1} = -q & \varepsilon^{2-21-1} = 1 & \varepsilon^{2-11-2} = -q^2 & \varepsilon^{21-1-2} = q^2 \\ \varepsilon^{-11-11} = k & \varepsilon^{1-11-1} = -k & \varepsilon^{ijkl} = 0 \text{ otherwise.} & \end{array}$$

For general N we can at least state the following properties, which can be easily proved as a consequence of (3.4):

Property. Let

$$I = \{i_1, \dots, i_N\}, \ J = \{1, \dots, N\}. \tag{3.14}$$

Then $\varepsilon^{i_1 \dots i_N} = 0$ unless all the following conditions are fulfilled:

1. if N is odd, the subset $J_0 = \{j \mid i_j = 0\}$ has an odd number of elements;
2. $J - J_0$ is an union of pairs $\{h, k\}$ such that $i_h = -i_k$;
3. the number $\natural_l$ of pairs $\{h, k\}$ such that $i_h = -i_k = l$ fulfils $\natural_l \leqslant n - l + 1$;
4. for no $j \in J - (J_0 \cup \{N\}) i_j = i_{j+1}$.

Property. [10]

$$g_{i_1 j_1} \dots g_{i_N j_N} \varepsilon^{j_N \dots j_1} =: \varepsilon_{i_1 i_2 \dots i_N} = \varepsilon^{i_N \dots i_2 i_1}. \tag{3.15}$$

We now give the relation connecting the antisymmetric projectors and the ε-tensors. In the appendix we prove

Proposition 3. *Let* $(d_0)^{-1} := \sum_{\{a_i\}} (\varepsilon^{a_1 \dots a_N})^2$. *Then*

$$\begin{aligned} \mathcal{P}^{-,l}{}^{i_1 \dots i_l}_{j_1 \dots j_l} &= d_l U^{k_1}_{j_1} \dots U^{k_l}_{j_l} \varepsilon^{i_{l+1} \dots i_N i_1 \dots i_l} \varepsilon^{i_{l+1} \dots i_N k_1 \dots k_l} \\ &\overset{(3.12)}{=} (-)^{l(N-1)} d_l \varepsilon^{j_1 \dots j_l i_{l+1} \dots i_N} \varepsilon^{i_{l+1} \dots i_N i_1 \dots i_l}, \end{aligned} \tag{3.16}$$

where d_l is defined by

$$\begin{aligned} d_l &:= d_0 \tfrac{[N]_q!}{[l]_q![N-l]_q!} && \text{if } H = U_q gl(N), \\ d_l &:= d_0 \tfrac{[N]_q!}{[l]_q![N-l]_q!} \tfrac{q^{l-\frac{N}{2}} + q^{\frac{N}{2}-l}}{q^{-\frac{N}{2}} + q^{\frac{N}{2}}} && \text{if } H = U_q so(N). \end{aligned} \tag{3.17}$$

Clearly $d_l = d_{N-l}$, in particular $d_0 = d_N$. In the $H = U_q so(N)$ case this can also be rewritten in the form

$$\mathcal{P}^{-,l}{}^{i_1\dots i_l}_{j_1\dots j_l} = d_l \varepsilon_{j_l\dots j_1}{}^{i_{l+1}\dots i_N} \varepsilon_{i_N\dots i_{l+1}}{}^{i_1\dots i_l}. \tag{3.18}$$

By an explicit calculation one finds that for the γ_3, γ_4 given above

$$d_0^{-1} = \begin{cases} [2]_{q^{1/2}}[3]_{q^{1/2}} & \text{if } N = 3 \\ 2([2]_{q^{1/2}})^2[3]_q & \text{if } N = 4. \end{cases} \tag{3.19}$$

4. Vielbein bases

The set of N exact forms $\{\xi^i\}$ is a natural basis for the F-bimodule Ω^1, as well as for the the $F \rtimes H$-bimodule $\Omega^1 \rtimes H$. By (1.11), the ξ^i *do not* commute with F. We are going to introduce alternative, so-called 'frames' (or 'vielbein' bases) [5] which *do*, revisiting the notion and construction given in [2].

As shown in [3, 12], there exists a algebra homomorphism

$$\varphi : \mathcal{A} \rtimes H \to \mathcal{A}, \tag{4.1}$$

acting as the identity on $\mathcal{A}$ itself,

$$\varphi(a) = a \qquad a \in \mathcal{A}, \tag{4.2}$$

where H is either Hopf algebra $H = U_q sl(N), U_q so(N)$ and $\mathcal{A} = \mathcal{H}$ is the corresponding deformed Heisenberg algebra on $\mathbb{C}_q^N, \mathbb{R}_q^N$. One can immediately extend φ to the central extensions $H = U_q gl(N), \widetilde{U_q so(N)}$ by setting

$$\varphi(q^{-2\eta}) = q^N \Lambda^{-2} \tag{4.3}$$

(adopting the same normalization factor q^N as in [13]), where the element $\Lambda^{-2} \in \mathcal{H}$ is defined by [20]

$$\Lambda^{-2} := 1 + qkx^i\partial_i \equiv 1 + O(h) \qquad \text{if } H = U_q sl(N), \tag{4.4}$$

$$\Lambda^{-2} := 1 + qkx^i\partial_i + \frac{q^N k^2}{(1+q^{N-2})^2} r^2 \Box \equiv 1 + O(h) \qquad \text{if } H = U_q so(N), \tag{4.5}$$

(in [20, 21] it was denoted by Λ). We are also extending $\mathcal{H}$ so as to contain its square root Λ^{-1} and inverse square root Λ as additional generators or as formal power series in the deformation parameter $h = \ln q$. The latter fulfil the relations

$$\Lambda x^i = q^{-1} x^i \Lambda, \qquad \Lambda \partial^i = q \partial^i \Lambda, \qquad \Lambda \xi^i = \xi^i \Lambda, \tag{4.6}$$

and the corresponding ones for Λ^{-1}. For real q, φ is even a $\star$-algebra homomorphism. Applying φ in particular to both sides of (1.16) one finds

$$a\varphi(g) = \varphi(g_{(1)})(a \triangleleft g_{(2)}). \tag{4.7}$$

In [2] to introduce a frame on $\mathbb{R}_q^N$ first auxiliary objects

$$\vartheta^i := q^{-\eta} \mathcal{L}^{-,i}_{\ \ l} \xi^l \in \Omega^1 \rtimes H \tag{4.8}$$

(with $H = \widetilde{U_q^- so(N)}$, the negative Borel subalgebra of $\widetilde{U_q so(N)}$) were introduced, characterized by the property to commute with F

$$[\vartheta^i, F] = 0. \tag{4.9}$$

The reader can check (4.9) by a direct computation that $[\vartheta^i, x^j] = 0$. In [2] we also showed that there exists a suitable map φ^- of the type (4.1), (4.2), with $\mathcal{A}$ a slight extension of F and $H = \widetilde{U_q^- so(N)}$. Replacing $q^{-\eta}\mathcal{L}^{-,i}_{\ l}$ by its φ^--image has no effect on the commutation relation with x^j, see (4.7), whence we found that the elements $\tilde{\theta}^i := \varphi^-\big(q^{-\eta}\mathcal{L}^{-,i}_{\ l}\big)\xi^l$ (in [2] denoted simply as $\{\theta^i\}$) also fulfilled (4.9), and therefore were called elements of a 'frame' (or 'vielbein'), according to the notion introduced in [5]. Now it is also easy to check that the *same* ϑ^i also commute with the derivatives, $[\vartheta^i, \partial^j] = 0$. By the same reasoning, replacing in the theorems and proofs of [2] the map φ^- with the one φ, we arrive at

Proposition 4. *The sets $\{\vartheta^i\}$ and $\{\theta^i\}$ of 1-forms given by*

$$\vartheta^i := q^{-\eta-\frac{N}{2}}\mathcal{L}^{-,i}_{\ l}\xi^l = \xi^m q^{1-\eta}\rho^j_m(u_4)\mathcal{L}^{-,i}_{\ j}, \tag{4.10}$$

$$\theta^i := \Lambda^{-1}\varphi\big(\mathcal{L}^{-,i}_{\ l}\big)\xi^l = \xi^h\Lambda^{-1}\varphi\big(S^2\mathcal{L}^{-,i}_{\ h}\big) \tag{4.11}$$

($u_4 := \mathcal{R}^{-1(1)}S^{-1}\mathcal{R}^{-1(2)}$) *are resp. 'frame' bases of the $\mathcal{H} \rtimes \widetilde{U_q so(N)}$-bimodule $\mathcal{DC}^1 \rtimes \widetilde{U_q so(N)}$ and of the $\mathcal{H}$-bimodule $\mathcal{DC}^1$, in the sense that*

$$[\vartheta^i, \mathcal{H}] = 0 \qquad [\theta^i, \mathcal{H}] = 0. \tag{4.12}$$

They satisfy the same commutation relations as the ξ^i,

$$(\mathbf{1}_{N^2} - \mathcal{P}^t)^{ij}_{hk}\vartheta^h\vartheta^k = 0 \qquad (\mathbf{1}_{N^2} - \mathcal{P}^t)^{ij}_{hk}\theta^h\theta^k = 0. \tag{4.13}$$

Finally, they form an N-plet under the action of $U_q^{op}so(N)$ (i.e. $U_q so(N)$ endowed with the opposite coproduct).

We just give the proof of the second equality in (4.11), which was not given in [2]. Recalling the coproduct $\Delta\big(\mathcal{L}^{-,i}_{\ k}\big) = \mathcal{L}^{-,i}_{\ h} \otimes \mathcal{L}^{-,h}_{\ k}$ of the FRT generators we find

$$\theta^i = \varphi\big(S\mathcal{L}^{-,h}_{\ k}S^2\mathcal{L}^{-,i}_{\ h}\big)\theta^k \overset{(4.12)_2}{=} \varphi\big(S\mathcal{L}^{-,h}_{\ k}\big)\theta^k\varphi\big(S^2\mathcal{L}^{-,i}_{\ h}\big)$$
$$\overset{(4.11)}{=} \xi^h\Lambda^{-1}\varphi\big(S^2\mathcal{L}^{-,i}_{\ h}\big) = \xi^h\Lambda^{-1}U^j_h\varphi\big(\mathcal{L}^{-,k}_{\ j}\big)U^{-1\,i}_{\ k}.$$

An analogous proposition for objects $\hat{\vartheta}^a, \hat{\theta}^a$ obtained by replacing ξ^i by $\hat{\xi}^i$, $\mathcal{L}^{-,i}_{\ h}$ by $\mathcal{L}^{+,i}_{\ h}$ and η by $-\eta$ holds. In [13] we have shown that the frame basis elements $\hat{\theta}^a$ transform exactly as the coordinates x^a under the $\star$-structure characterizing real q (namely can be made real by a suitable $\mathbb{C}$-linear transformation).

Explicit expressions for the images $\varphi\big(\mathcal{L}^{-,h}_{\ k}\big)$ for the H with the lowest N, $H = U_q sl(2), U_q so(3)$, can be found e.g. in [14], section 4.

As the commutation relations (4.13) among the θ^h are exactly of the same form of the ones (1.12) among the ξ^i, we immediately find that also the space $\bigwedge^N_\theta$ of monomials in θ^i of degree N has dimension 1. Moreover,

$$\theta^{i_1}\theta^{i_2}\ldots\theta^{i_N} = \mathrm{d}V\,\varepsilon^{i_1 i_2\ldots i_N}, \tag{4.14}$$

where $\mathrm{d}V \in \bigwedge^N_\theta$ is defined replacing in (3.5) $\mathrm{d}^N x$ with $\mathrm{d}V$ and ξ^i with θ^i.

Proposition 5. *The 'volume form' $\mathrm{d}V$ is central in $\mathcal{DC}^*$ and equal to*

$$\mathrm{d}V = \mathrm{d}^N x\,\Lambda^{-N}. \tag{4.15}$$

Proof. With the definition of $\mathrm{d}V$ adopted for $H = \widetilde{U_q so(N)}$ (the case $H = U_q gl(N)$ is completely analogous)

$$\begin{aligned}\mathrm{d}V &\overset{(4.11)}{=} \theta^{-n}\dots\theta^{n-1}\Lambda^{-1}\varphi\big(\mathcal{L}^{-,n}_{\ i_N}\big)\xi^{i_N} \overset{(4.12)_2}{=} \Lambda^{-1}\varphi\big(\mathcal{L}^{-,n}_{\ i_N}\big)\theta^{-n}\dots\theta^{n-1}\xi^{i_N}\\ &= \cdots = \Lambda^{-1}\varphi\big(\mathcal{L}^{-,n}_{\ i_N}\big)\dots\Lambda^{-1}\varphi\big(\mathcal{L}^{-,-n}_{\ i_1}\big)\xi^{i_1}\dots\xi^{i_N}\\ &\overset{(3.6)}{=} \Lambda^{-N}\varphi\big(\mathcal{L}^{-,n}_{\ i_N}\dots\mathcal{L}^{-,-n}_{\ i_1}\big)\varepsilon^{i_1\dots i_N}\,\mathrm{d}^N x \overset{(3.8),(4.3)}{=} \Lambda^{-N}\,\mathrm{d}^N x.\end{aligned}$$

□

The reader might wonder about the usefulness of the generalized notion of vielbein introduced in this section: generally speaking the differential forms $\omega_p \in \Omega^p$ and the functions $f \in F$ have a geometrical or physical significance, so since θ^a are in $\mathcal{DC}^1$ rather than in Ω^1, the components of ω_p in the vielbein basis are in $\mathcal{H}$ rather than in F. The point is that, as we have shown in [13], the difference between these components is irrelevant when evaluating functionals on Ω^N, scalar products in Ω^p, etc by means of integration, provided Stokes' theorem applies.

5. Hodge map and Laplacian on $\mathbb{R}^N_q$

Having at one's disposal also the $U_q so(N)$-covariant metric matrix g_{ij}, a ($U_q so(N)$-covariant) Hodge map $* : \bigwedge^p \to \bigwedge^{N-p}$ acting on *exterior* forms on $\mathbb{R}^N_q$ was introduced (leaving some ambiguities) in [11, 18] using both g_{ij} and the q-epsilon tensor, in analogy with the undeformed theory. As we are going to see, one has to fix the ambiguities to make $*$ involutive and moreover add in the definition a suitable power of Λ in order to define a Hodge map on *differential* forms. It is more convenient to start giving the definition of the Hodge map in the frame basis:

Proposition 6. *For $H = U_q so(N)$ and any $p = 0, 1, \dots, N$ one can define a $\mathcal{H}$-bilinear map*

$$* : \mathcal{DC}^p \to \mathcal{DC}^{N-p} \tag{5.1}$$

the 'Hodge map', such that ${}^*\mathbf{1} = \mathrm{d}V$ and on each $\mathcal{DC}^p$, and therefore on the whole $\mathcal{DC}^*$,

$$*^2 \equiv * \circ * = id \tag{5.2}$$

by setting on the monomials in the θ^a

$${}^*(\theta^{a_1}\theta^{a_2}\dots\theta^{a_p}) = c_p\theta^{a_{p+1}}\dots\theta^{a_N}\varepsilon_{a_N\dots a_{p+1}}{}^{a_1\dots a_p}; \tag{5.3}$$

the normalization constants c_p are constrained by the conditions

$$c_p c_{N-p} = d_p. \tag{5.4}$$

The most convenient choice for the c_p will be given below. $\mathcal{H}$-bilinearity implies in particular

$${}^*(a\omega_p b) = a^*\omega_p b \qquad \forall\, a, b \in \mathcal{H}, \quad \omega_p \in \mathcal{DC}^p; \tag{5.5}$$

i.e. applying Hodge and multiplying by 'functions or differential operators' are commuting operations, in other words a differential form ω_p and its Hodge map image have the same commutation relations with x^i, ∂^j. That this is true is evident in the frame basis, because of (4.12). Relation (5.2) easily follows from (3.18). The fixed positive sign on the rhs of (5.2) (cumbersome when compared to the more familiar $(-1)^{p(N-p)}$) is the sign of d_0 and is due to the non-standard ordering of the indices in (5.3). The latter in turn is the only correct one: had we used a different order, on the rhs of (5.2) tensor products of the matrices $U^{\pm 1}$, instead of the identity, would have appeared, because of property (3.12).

Using the $\mathcal{H}$-bilinearity of $*$ in the appendix we prove

Proposition 7. *In terms of the basis of differentials (5.3) takes the form*

$$*(\xi^{i_1}\dots\xi^{i_p}) = q^{-N(p-N/2)}c_p\xi^{i_{p+1}}\dots\xi^{i_N}\varepsilon_{i_N\dots i_{p+1}}{}^{i_1\dots i_p}\Lambda^{2p-N}. \tag{5.6}$$

This differs from the (incomplete) definition of Hodge map on exterior forms given in [11, 18] by the presence of Λ-powers (needed for the $\mathcal{H}$-bilinearity), by the already noted crucial different indices order and by the explicit determination of the coefficients c_p. From the above formulae and the commutation relations (1.11), (4.6) it is evident that by restricting the domain of $*$ to the unital subalgebra $\widetilde{\Omega}^* \subset \mathcal{DC}^*$ generated by x^i, ξ^j, $\Lambda^{\pm 1}$ one obtains a $\widetilde{F}$-bilinear map

$$* : \widetilde{\Omega}^p \to \widetilde{\Omega}^{N-p} \tag{5.7}$$

fulfilling again (5.2) (just take $a, b \in \widetilde{F}$ in (5.5)); here $\widetilde{F}$ denotes the unital subalgebra generated by x^i, $\Lambda^{\pm 1}$. This restriction is what is the notion closest to the conventional notion of a Hodge map on $\mathbb{R}^N_q$: as a matter of fact, there is no F-bilinear restriction of $*$ to Ω^*.

From the bilinearity of the Hodge map and the explicit $U_q so(N)$-covariant form of (5.6) it immediately follows

Proposition 8. *The Hodge map is $U_q so(N)$-covariant, i.e. commutes with the $U_q so(N)$-action:*

$$({}^*\omega_p) \triangleleft g = {}^*(\omega_p \triangleleft g) \qquad \forall\, g \in U_q so(N). \tag{5.8}$$

This is true also for its restriction to the subalgebra $\widetilde{\Omega}^* \subset \mathcal{DC}^*$.

Remark. But $*$ is not $\widetilde{U_q so(N)}$-covariant. This is due to the fact that η has a nontrivial action on each ξ^i, and $*$ changes the degree of a monomial in the ξ^i.

As in commutative geometry we introduce the exterior coderivative by

$$\delta := -{}^*\mathrm{d}^*. \tag{5.9}$$

In [13] we show that (at least for positive q) δ can be seen as the Hermitian conjugate of d acting on Ω^* endowed with a suitable scalar product. The residual freedom left by (5.4) in choosing the c_p is eliminated by requiring that the differential operator $\mathrm{d}\delta + \delta\mathrm{d}$ is a scalar proportional to $\partial\cdot\partial$, as in the commutative geometry case. In the appendix we prove the following proposition:

Proposition 9. *The 'Laplacian' $\Delta := \mathrm{d}\delta + \delta\mathrm{d}$ reduces on all $\mathcal{DC}^*$, and in particular on Ω^*, to*

$$\Delta = -q^2\partial\cdot\partial\Lambda^2 = -q^{-N}\hat{\partial}\cdot\hat{\partial}, \tag{5.10}$$

provided we choose

$$c_p = \frac{1}{[N-p]_q!}\prod_{l=p}^{N-1}\frac{q^{l-\frac{N}{2}} + q^{\frac{N}{2}-l}}{q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1}}. \tag{5.11}$$

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Appendix

We begin this appendix by recalling a few basic properties about the universal R-matrix, or quasitriangular structure [6], $\mathcal{R}$ of the quantum groups $U_q\mathbf{g}$, while fixing our conventions. $\mathcal{R}$ intertwines between Δ and opposite coproduct Δ^{op}, and so does also $\mathcal{R}_{21}^{-1}$:

$$\mathcal{R}(g_{(1)} \otimes g_{(2)}) = (g_{(2)} \otimes g_{(1)})\mathcal{R}, \qquad \mathcal{R}_{21}^{-1}(g_{(1)} \otimes g_{(2)}) = (g_{(2)} \otimes g_{(1)})\mathcal{R}_{21}^{-1}. \tag{A.1}$$

It fulfils

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{A.2}$$

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{A.3}$$

$$(S \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{-1} = (\mathrm{id} \otimes S^{-1})\mathcal{R}, \tag{A.4}$$

$$S^{-1}(g) = u^{-1}S(g)u, \tag{A.5}$$

where u, which is defined up to an invertible central factor, can be taken e.g. as the $u = u_1$ with

$$u_1 := (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}, \tag{A.6}$$

From (A.1)–(A.3) it follows the universal Yang–Baxter relation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{A.7}$$

The braid matrix $\hat{R}$ [8] is related to the quasitriangular structure $\mathcal{R}$ by $\hat{R}^{ij}_{hk} \equiv R^{ji}_{hk} := q^{\sigma_N}\big(\rho^j_h \otimes \rho^i_k\big)\mathcal{R} = q^{\sigma_N}\rho^j_h\big(\mathcal{L}^{+,i}{}_k\big)$, where $\sigma_N = 1/N$ for $H = U_q gl(N)$ and $\sigma_N = 0$ for $H = U_q so(N)$. With the indices' convention described in section 3, $\hat{R}$ is given by

$$\hat{R} = q\sum_i e^i_i \otimes e^i_i + \sum_{i\neq j} e^j_i \otimes e^i_j + k\sum_{i<j} e^i_i \otimes e^j_j \tag{A.8}$$

when $H = U_q sl(N)$, and by

$$\begin{aligned}\hat{R} = {} & q\sum_{i\neq 0} e^i_i \otimes e^i_i + \sum_{\substack{i\neq j,-j \\ \text{or } i=j=0}} e^j_i \otimes e^i_j + q^{-1}\sum_{i\neq 0} e^{-i}_i \otimes e^i_{-i} \\ & + k\left(\sum_{i<j} e^i_i \otimes e^j_j - \sum_{i<j} q^{-\rho_i+\rho_j} e^{-j}_i \otimes e^j_{-i}\right)\end{aligned} \tag{A.9}$$

when $H = U_q so(N)$. Here e^i_j is the $N \times N$ matrix with all elements equal to zero except for a 1 in the ith column and jth row, and $k := q - q^{-1}$.

In the $H = U_q so(N)$ case, using (1.7), (1.5) it is not difficult to show the following formulae,

$$\begin{aligned}\mathcal{P}^t_{12}\hat{R}^{\pm 1}_{23} &= Q_N\mathcal{P}^t_{12}\mathcal{P}^t_{23}\hat{R}^{\mp 1}_{12}, \qquad & \hat{R}^{\pm 1}_{23}\mathcal{P}^t_{12} &= Q_N\hat{R}^{\mp 1}_{12}\mathcal{P}^t_{23}\mathcal{P}^t_{12}, \\ \mathcal{P}^t_{23}\hat{R}^{\pm 1}_{12} &= Q_N\mathcal{P}^t_{23}\mathcal{P}^t_{12}\hat{R}^{\mp 1}_{23}, \qquad & \hat{R}^{\pm 1}_{12}\mathcal{P}^t_{23} &= Q_N\hat{R}^{\mp 1}_{23}\mathcal{P}^t_{12}\mathcal{P}^t_{23},\end{aligned} \tag{A.10}$$

which are written in matrix-tensor notation in order to let us do many proofs avoiding indices i, j, etc. Moreover,

$$\rho^a_b(Sh) = g^{ad}\rho^c_d(h)g_{cb}, \quad \Rightarrow \quad S\mathcal{L}^{\mp,j}_{\ i} = g_{ih}\mathcal{L}^{\mp,h}_{\ k}g^{kj}. \tag{A.11}$$

Proof of proposition 1

One can determine the projectors $\mathcal{P}^{\pm,l}$ iteratively. We adopt the ansatz (2.4) with $M^{\pm} = f^{\pm}(\hat{R})$ a matrix to be determined. The most general one is

$$\begin{aligned} M^{\pm,l+1} &= \alpha^{\pm}_{l_{N^2}+1}\left(\mathbf{1} + \beta^{\pm}_{l+1}\hat{R}\right) && \text{if } H = U_q sl(N) \\ M^{\pm,l+1} &= \alpha^{\pm}_{l_{N^2}+1}\left(\mathbf{1} + \beta^{\pm}_{l+1}\hat{R} + \gamma^{\pm}_{l+1}\mathcal{P}^{t}\right) && \text{if } H = U_q so(N). \end{aligned} \tag{A.12}$$

We first determine the coefficients $\beta^{\pm}_{l+1}$, $\gamma^{\pm}_{l+1}$ by imposing the conditions (2.1). By the recursive assumption, only the condition with $m = l$ is not fulfiled automatically and must be imposed by hand. Actually, it suffices to impose just $(2.1)_1$, due to the symmetry of the ansatz (2.4) and of the matrices $\mathcal{P}^{\pi}$ under transposition. Setting

$$\begin{aligned} \mathcal{P}'^{\mp}_{l(l+1)} &:= \mathcal{P}^{\mp}_{l(l+1)} && \text{if } H = U_q sl(N), \\ \mathcal{P}'^{\mp}_{l(l+1)} &:= \mathcal{P}^{\mp}_{l(l+1)} + \mathcal{P}^{t}_{l(l+1)} && \text{if } H = U_q so(N), \end{aligned}$$

this amounts to

$$\begin{aligned} 0 \stackrel{!}{=} \mathcal{P}^{\pm,l+1}\mathcal{P}'^{\mp}_{l(l+1)} &\stackrel{(2.4)}{=} \mathcal{P}^{\pm,l}_{1\dots l} M^{\pm}_{l(l+1)} \mathcal{P}^{\pm,l-1}_{1\dots(l-1)} M^{\pm}_{(l-1)l} \mathcal{P}^{\pm,l-1}_{1\dots(l-1)} \mathcal{P}'^{\mp}_{l(l+1)} \\ &= \mathcal{P}^{\pm,l}_{1\dots l} M^{\pm}_{l(l+1)} M^{\pm}_{(l-1)l} \mathcal{P}^{\pm,l-1}_{1\dots(l-1)} \mathcal{P}'^{\mp}_{l(l+1)}. \end{aligned} \tag{A.13}$$

In the $H = U_q sl(N)$ case (A.13) becomes

$$\begin{aligned} 0 &\propto \mathcal{P}^{\pm,l}_{1\dots l}\left[\mathbf{1}_{N^{l+1}} + \beta^{\pm}_{l+1}\hat{R}_{l(l+1)} + \beta^{\pm}_{l}\hat{R}_{(l-1)l} + \beta^{\pm}_{l}\beta^{\pm}_{l+1}\hat{R}_{l(l+1)}\hat{R}_{(l-1)l}\right]\mathcal{P}^{\pm,l-1}_{1\dots(l-1)}\mathcal{P}'^{\mp}_{l(l+1)} \\ &= \mathcal{P}^{\pm,l}_{1\dots l}\left[1 \mp q^{\mp 1}\beta^{\pm}_{l+1} \pm q^{\pm 1}\beta^{\pm}_{l}\right]\mathcal{P}^{\pm,l-1}_{1\dots(l-1)}\mathcal{P}'^{\mp}_{l(l+1)}, \end{aligned} \tag{A.14}$$

where we have used the braid equation (1.4) to see that the term proportional to $\beta^{\pm}_{l}\beta^{\pm}_{l+1}$ vanishes, and the relations

$$\hat{R}\mathcal{P}'^{\mp} = \mp q^{\mp 1}\mathbf{1}_{N^2}\mathcal{P}'^{\mp}, \qquad \mathcal{P}^{\pm,l}_{1\dots l}\hat{R}_{(l-1)l} = \pm q^{\pm 1}\mathcal{P}^{\pm,l}_{1\dots l}.$$

The condition that the square bracket in (A.14) vanishes is recursively solved, starting from $l = 1$ with initial input $\beta^{\pm}_{1} = 0$ (since $\mathcal{P}^{\pm,1} = \mathbf{1}_N$), by

$$\beta^{\pm}_{l+1} = \pm q^{\pm 1} l_{q^{\pm 2}} \tag{A.15}$$

(for $l = 2$ this gives back (1.8)).

In the $H = U_q so(N)$ case (A.13) becomes

$$\begin{aligned} 0 \propto \mathcal{P}^{\pm,l}_{1\dots l} &\left[\mathbf{1}_{N^{l+1}} + \beta^{\pm}_{l+1}\hat{R}_{l(l+1)} + \gamma^{\pm}_{l+1}\mathcal{P}^{t}_{l(l+1)} + \beta^{\pm}_{l}\hat{R}_{(l-1)l} + \gamma^{\pm}_{l}\mathcal{P}^{t}_{(l-1)l}\right. \\ &+ \beta^{\pm}_{l}\beta^{\pm}_{l+1}\hat{R}_{l(l+1)}\hat{R}_{(l-1)l} + \gamma^{\pm}_{l}\gamma^{\pm}_{l+1}\mathcal{P}^{t}_{l(l+1)}\mathcal{P}^{t}_{(l-1)l} + \beta^{\pm}_{l}\gamma^{\pm}_{l+1}\mathcal{P}^{t}_{l(l+1)}\hat{R}_{(l-1)l} \\ &\left.+ \gamma^{\pm}_{l}\beta^{\pm}_{l+1}\hat{R}_{l(l+1)}\mathcal{P}^{t}_{(l-1)l}\right]\mathcal{P}^{\pm,l-1}_{1\dots(l-1)}\mathcal{P}'^{\mp}_{l(l+1)} \\ = \mathcal{P}^{\pm,l}_{1\dots l} &\left\{\mathbf{1}_{N^{l+1}} + \beta^{\pm}_{l+1}\left[\mp q^{\mp 1}\mathbf{1}_{N^{l+1}} + (q^{1-N} \pm q^{\mp 1})\mathcal{P}^{t}_{l(l+1)}\right] + \gamma^{\pm}_{l+1}\mathcal{P}^{t}_{l(l+1)}\right. \\ &\pm q^{\pm 1}\beta^{\pm}_{l}\mathbf{1}_{N^{l+1}} + \gamma^{\pm}_{l}\gamma^{\pm}_{l+1}\mathcal{P}^{t}_{l(l+1)}\mathcal{P}^{t}_{(l-1)l} + \beta^{\pm}_{l}\gamma^{\pm}_{l+1}Q_N\mathcal{P}^{t}_{l(l+1)}\mathcal{P}^{t}_{(l-1)l}\left[\mp q^{\pm 1}\mathbf{1}_{N^{l+1}}\right. \\ &\left.\left.+ (q^{N-1} \pm q^{\pm 1})\mathcal{P}^{t}_{l(l+1)}\right] \pm q^{\mp 1}\gamma^{\pm}_{l}\beta^{\pm}_{l+1}Q_N\mathcal{P}^{t}_{l(l+1)}\mathcal{P}^{t}_{(l-1)l}\right\}\mathcal{P}^{\pm,l-1}_{1\dots(l-1)}\mathcal{P}'^{\mp}_{l(l+1)} \\ = \mathcal{P}^{\pm,l}_{1\dots l} &\left\{\mathbf{1}_{N^{l+1}}\left[1 \mp q^{\mp 1}\beta^{\pm}_{l+1} \pm q^{\pm 1}\beta^{\pm}_{l}\right] + \mathcal{P}^{t}_{l(l+1)}\left[\beta^{\pm}_{l+1}(q^{1-N} \pm q^{\mp 1}) + \gamma^{\pm}_{l+1}\right.\right. \\ &\left.+ \beta^{\pm}_{l}\gamma^{\pm}_{l+1}\frac{q^{N-1} \pm q^{\pm 1}}{Q_N}\right] + \mathcal{P}^{t}_{l(l+1)}\mathcal{P}^{t}_{(l-1)l}\left[\gamma^{\pm}_{l}\gamma^{\pm}_{l+1} \mp \beta^{\pm}_{l}\gamma^{\pm}_{l+1}Q_N q^{\pm 1}\right. \\ &\left.\left.\pm q^{\mp 1}\gamma^{\pm}_{l}\beta^{\pm}_{l+1}Q_N\right]\right\}\mathcal{P}^{\pm,l-1}_{1\dots(l-1)}\mathcal{P}'^{\mp}_{l(l+1)}, \end{aligned}$$

where we have used the braid equation (1.4) to see that the term proportional to $\beta_l^\pm\beta_{l+1}^\pm$ vanishes, and the relations

$$\hat{R}\mathcal{P}'^{\mp} = [\mp q^{\mp 1}\mathbf{1}_{N^2} + (q^{1-N} \pm q^{\mp 1})\mathcal{P}^t]\mathcal{P}'^{\mp}$$
$$\begin{aligned}\mathcal{P}^t_{l(l+1)}\hat{R}_{(l-1)l}\mathcal{P}'^{\mp}_{l(l+1)} &= Q_N\mathcal{P}^t_{l(l+1)}\mathcal{P}^t_{(l-1)l}\hat{R}^{-1}_{l(l+1)}\mathcal{P}'^{\mp}_{l(l+1)}\\ &= Q_N\mathcal{P}^t_{l(l+1)}\mathcal{P}^t_{(l-1)l}\left[\mp q^{\pm 1}\mathbf{1}_{N^{l+1}} + (q^{N-1} \pm q^{\pm 1})\mathcal{P}^t_{l(l+1)}\right]\mathcal{P}'^{\mp}_{l(l+1)}\end{aligned}$$
$$\mathcal{P}^{\pm,l}_{1\dots l}\hat{R}_{l(l+1)}\mathcal{P}^t_{(l-1)l} = Q_N\mathcal{P}^{\pm,l}_{1\dots l}\hat{R}^{-1}_{(l-1)l}\mathcal{P}^t_{l(l+1)}\mathcal{P}^t_{(l-1)l} = \pm q^{\mp 1}Q_N\mathcal{P}^{\pm,l}_{1\dots l}\mathcal{P}^t_{l(l+1)}\mathcal{P}^t_{(l-1)l}$$
$$\mathcal{P}^t_{l(l+1)}\mathcal{P}^t_{(l-1)l}\mathcal{P}^t_{l(l+1)} = \frac{1}{Q_N^2}\mathcal{P}^t_{l(l+1)}.$$

The conditions that the three square brackets vanish

$$1 \mp q^{\mp 1}\beta^\pm_{l+1} \pm q^{\pm 1}\beta^\pm_l = 0,$$
$$\beta^\pm_{l+1}(q^{1-N} \pm q^{\mp 1}) + \gamma^\pm_{l+1} + \beta^\pm_l\gamma^\pm_{l+1}\frac{q^{N-1} \pm q^{\pm 1}}{Q_N} = 0,$$
$$\gamma^\pm_l\gamma^\pm_{l+1} \mp q^{\pm 1}\beta^\pm_l\gamma^\pm_{l+1}Q_N \pm q^{\mp 1}\gamma^\pm_l\beta^\pm_{l+1}Q_N = 0,$$

are recursively solved, starting from $l = 1$ with initial input $\beta^\pm_1 = 0 = \gamma^\pm_1$ (since $\mathcal{P}^{\pm,1} = \mathbf{1}_N$), again by (A.15) and by

$$\gamma^+_{l+1} = \frac{(q^N-1)(1+q^{2-N})}{1-q^{N+2l-2}}l_{q^2} \qquad \gamma^-_{l+1} = \frac{(q^{-N}-1)(1+q^{N-2})}{1-q^{N-2l}}l_{q^{-2}} \tag{A.16}$$

(for $l = 2$ this gives back (1.9)).

We determine the coefficient $\alpha^\pm_{l+1}$ by imposing condition (2.2). For both $H = U_qsl(N)$, $U_qso(N)$ this gives

$$\begin{aligned}0 &\stackrel{!}{=} \mathcal{P}^{\pm,l+1}(\mathcal{P}^{\pm,l+1} - \mathbf{1}_{N^{l+1}}) \stackrel{(2.4)}{=} \mathcal{P}^{\pm,l+1}\left(\mathcal{P}^{\pm,l}_{1\dots l}M^\pm_{l(l+1)}\mathcal{P}^{\pm,l}_{1\dots l} - \mathbf{1}_{N^{l+1}}\right)\\ &= \mathcal{P}^{\pm,(l+1)}\left[\alpha^\pm_{l+1}\left(1 \pm q^{\pm 1}\beta^\pm_{l+1}\right) - 1\right];\end{aligned}$$

in the last equality we have used (2.1), (A.12), (1.1). The condition that the square bracket vanishes is recursively solved, starting from $l = 0$ with initial input $\alpha^\pm_0 = 1$, by

$$\alpha^\pm_{l+1} = \frac{1}{(l+1)_{q^{\pm 2}}}.$$

By using (1.6) we give at the form (2.6) for $M^{\pm,l+1}$.

To check that (2.3) is satisfied we just note that the dimension of each projector is an integer, that it is the required one for $q = 1$ (since in this limit the projector reduces to its undeformed counterpart), and therefore it is also for any generic q, by continuity in q.

Proof of proposition 2

Being H-invariant, the element $\xi^{i_1}\xi^{i_2}\dots\xi^{i_N} \in \bigwedge^N$ commutes with all H (within $\mathcal{DC}^* \rtimes H$). Therefore

$$\begin{aligned}\xi^{i_1}\dots\xi^{i_N} &\stackrel{(4.6)}{=} u_1^{-1}(S\mathcal{R}^{(2)})\xi^{i_1}\dots\xi^{i_N}\mathcal{R}^{(1)}\\ &\stackrel{(1.16)}{=} u_1^{-1}(S\mathcal{R}^{(2)})\mathcal{R}^{(1)}_{(1)}\left(\xi^{i_1} \triangleleft \mathcal{R}^{(1)}_{(2)}\right)\dots\left(\xi^{i_N} \triangleleft \mathcal{R}^{(1)}_{(N+1)}\right)\\ &\stackrel{(1.17)}{=} u_1^{-1}(S\mathcal{R}^{(2)})\mathcal{R}^{(1)}_{(1)}\rho^{i_1}_{j_1}\left(\mathcal{R}^{(1)}_{(2)}\right)\dots\rho^{i_N}_{j_N}\left(\mathcal{R}^{(1)}_{(N+1)}\right)\xi^{j_1}\dots\xi^{j_N}\\ &\stackrel{(A.2)}{=} u_1^{-1}\left[S\mathcal{R}^{(2)}_N\right]\dots\left[S\mathcal{R}^{(2)}_1\right]\left[S\mathcal{R}^{(2)}\right]\mathcal{R}^{(1)}\rho^{i_1}_{j_1}\left[\mathcal{R}^{(1)}_1\right]\dots\rho^{i_N}_{j_N}\left[\mathcal{R}^{(1)}_N\right]\xi^{j_1}\dots\xi^{j_N}\end{aligned}$$

$$\stackrel{(A.5)}{=} \left(S^{-1}\mathcal{R}_N^{(2)}\right)\ldots\left(S^{-1}\mathcal{R}_1^{(2)}\right)\rho_{j_1}^{i_1}\left(\mathcal{R}_1^{(1)}\right)\ldots\rho_{j_N}^{i_N}\left(\mathcal{R}_N^{(1)}\right)\xi^{j_1}\ldots\xi^{j_N}$$
$$\stackrel{(A.4)}{=} \mathcal{R}_N^{-1(2)}\ldots\mathcal{R}_1^{-1(2)}\rho_{j_1}^{i_1}\left(\mathcal{R}_1^{-1(1)}\right)\ldots\rho_{j_N}^{i_N}\left(\mathcal{R}_N^{-1(1)}\right)\xi^{j_1}\ldots\xi^{j_N}$$
$$\stackrel{(3.7)}{=} \mathcal{L}^{-,i_N}_{\ j_N}\ldots\mathcal{L}^{-,i_1}_{\ j_1}\xi^{j_1}\ldots\xi^{j_N},$$

where $\mathcal{R}_1,\ldots,\mathcal{R}_N$ just denote N different copies of $\mathcal{R}$; factoring out $\mathrm{d}^N x$ (see equation (3.6)) the claim follows.

Proof of proposition 3

We start by recalling the relations (which can be easily checked using the explicit definition of $\hat{R}, U, \mathcal{P}^t$ given above)

$$\mathrm{tr}_2(U_2\hat{R}_{12}) = q^N\mathbf{1}_N \qquad \mathrm{tr}(U) = [N]_q \qquad \text{if } H = U_q gl(N)$$
$$\mathrm{tr}_2(U_2\hat{R}_{12}) = q^{N-1}\mathbf{1}_N \qquad \mathrm{tr}_2\left(U_2\mathcal{P}_{12}^t\right) = \frac{\mathbf{1}_N}{Q_N} \qquad \mathrm{tr}(U) = Q_N, \quad \text{if } H = U_q so(N),$$

where $U_2^{\pm 1} \equiv \mathbf{1}_N \otimes U^{\pm 1}$ and tr_2 denotes matrix trace on the second factor in the tensor product $\mathbb{C}^N \otimes \mathbb{C}^N$; this implies

$$\mathrm{tr}_2\left(U_2 M_{12}^{-,l}\right) = b_l\mathbf{1}_N, \tag{A.17}$$

where

$$b_l = \frac{[N-l+1]_q}{[l]_q} \qquad \text{if } H = U_q gl(N) \tag{A.18}$$
$$b_l = \frac{1}{[l]_q}\left[q^{l-1}Q_N - [l-1]_q q^{N-1} + \frac{(q^{-2}-1)[l-1]_q}{q^{-1}+q^{N+1-2l}}\right]$$
$$= \frac{[N-l+1]_q}{[l]_q}\frac{q^{\frac{N}{2}-l}+q^{l-\frac{N}{2}}}{q^{\frac{N}{2}+1-l}+q^{l-1-\frac{N}{2}}} \qquad \text{if } H = U_q so(N). \tag{A.19}$$

By definition (3.6) of the ε-tensor and (1.12) the claim is manifestly true for $l = N$,

$$\mathcal{P}^{-,N}_{\ j_1\ldots j_N}{}^{i_1\ldots i_N} = d_N\varepsilon^{i_1\ldots i_N}\varepsilon^{j_1\ldots j_N},$$

because the rhs fulfils all conditions (2.1)–(2.3). We prove the claim for the remaining $l < N$ by induction, with N inductive steps. Assume the claim is true for $l = m+1$:

$$\mathcal{P}^{-,m+1}_{\ j_1\ldots j_{m+1}}{}^{i_1\ldots i_{m+1}} = d_{m+1}U_{j_1}^{k_1}\ldots U_{j_{m+1}}^{k_{m+1}}\varepsilon^{i_{m+2}\ldots i_N i_1\ldots i_{m+1}}\varepsilon^{i_{m+2}\ldots i_N k_1\ldots k_{m+1}}.$$

Multiplying both sides by $U_{i_{m+1}}^{j_{m+1}}$ (and summing of course also on the repeated indices i_{m+1}, j_{m+1}) we find on one hand

$$[\mathrm{tr}_{m+1}(U_{m+1}\mathcal{P}^{-,m+1})]^{i_1\ldots i_m}_{j_1\ldots j_m}$$
$$= d_{m+1}U_{j_1}^{k_1}\ldots U_{j_m}^{k_m}U^{2k_{m+1}}_{\ i_{m+1}}\varepsilon^{i_{m+2}\ldots i_N i_1\ldots i_{m+1}}\varepsilon^{i_{m+2}\ldots i_N k_1\ldots k_{m+1}}$$
$$\stackrel{(3.12)}{=} d_{m+1}U_{j_1}^{k_1}\ldots U_{j_m}^{k_m}\varepsilon^{i_{m+1}\ldots i_N i_1\ldots i_m}\varepsilon^{i_{m+\ldots}i_N k_1\ldots k_m},$$

and on the other

$$\mathrm{tr}_{m+1}(U_{m+1}\mathcal{P}^{-,m+1}) \stackrel{(2.4)}{=} \mathrm{tr}_{m+1}\left(U_{m+1}\mathcal{P}^{-,m}_{12\ldots m}M^{-,m+1}_{m(m+1)}\mathcal{P}^{-,m}_{12\ldots l}\right)$$
$$\stackrel{(A.17)}{=} b_{m+1}\mathcal{P}^{-,m}_{12\ldots m},$$

whence by comparison the claim for $l = m$ follows, because $d_m b_{m+1} = d_{m+1}$.

Proof of proposition 7

$$
\begin{aligned}
\text{lhs}(5.6) &\overset{(4.11)}{=} {}^*\big[\varphi\big(q^{\eta} S\mathcal{L}^{-,i_1}_{\ j_1}\big)\theta^{j_1}..\varphi\big(q^{\eta} S\mathcal{L}^{-,i_p}_{\ j_p}\big)\theta^{j_p}\big]\\
&\overset{(4.12)}{=} {}^*\big[\varphi\big(q^{\eta} S\mathcal{L}^{-,i_1}_{\ j_1}\big)\dots\varphi\big(q^{\eta} S\mathcal{L}^{-,i_p}_{\ j_p}\big)\theta^{j_1}..\theta^{j_p}\big]\\
&\overset{(5.5)}{=} \varphi\big(q^{p\eta} S\big(\mathcal{L}^{-i_p}_{\ \ j_p}\dots\mathcal{L}^{-,i_1}_{\ j_1}\big)\big)^*[\theta^{j_1}..\theta^{j_p}]\\
&\overset{(5.3)}{=} c_p q^{-Np/2}\Lambda^p\varphi\big(S\big(\mathcal{L}^{-i_p}_{\ \ j_p}\dots\mathcal{L}^{-,i_1}_{\ j_1}\big)\big)\theta^{h_{p+1}}\dots\theta^{h_N}\varepsilon_{h_N\dots h_{p+1}}{}^{j_1\dots j_p}\\
&= c_p q^{-Np/2}\Lambda^p\varphi\big(\mathcal{L}^{-,l_{p+1}}_{\ \ j_{p+1}}\dots\mathcal{L}^{-,l_N}_{\ \ k_N}\big)\varphi\big(S\big(\mathcal{L}^{-i_p}_{\ \ j_p}\dots\mathcal{L}^{-,i_1}_{\ j_1}\mathcal{L}^{-,k_{p+1}}_{\ \ j_{p+1}}\dots\mathcal{L}^{-,k_N}_{\ \ j_N}\big)\\
&\quad\times\theta^{h_{p+1}}\dots\theta^{h_N}g_{h_N l_N}\dots g_{h_{p+1}l_{p+1}}\varepsilon^{j_N\dots j_{p+1}j_1\dots j_p}\\
&\overset{(3.8)}{=} c_p q^{-Np/2}\Lambda^p\varphi\big(\mathcal{L}^{-,l_{p+1}}_{\ \ j_{p+1}}\dots\mathcal{L}^{-,l_N}_{\ \ k_N}\big)\theta^{h_{p+1}}\dots\theta^{h_N}\\
&\quad\times g_{h_N l_N}\dots g_{h_{p+1}l_{p+1}}\varepsilon^{k_N\dots k_{p+1}i_1\dots i_p}\\
&\overset{(A.11)}{=} c_p q^{-Np/2}\Lambda^p g_{l_N j_N}\dots g_{j_{p+1}l_{p+1}}\varphi\big(\big(S\mathcal{L}^{-,l_{p+1}}_{\ \ h_{p+1}}\big)\dots\big(S\mathcal{L}^{-,l_N}_{\ \ h_N}\big)\big)\\
&\quad\times\theta^{h_{p+1}}\dots\theta^{h_N}\varepsilon^{k_N\dots k_{p+1}i_1\dots i_p}\\
&\overset{(4.11)}{=} c_p q^{-N(p-N/2)}\Lambda^{2p-N}g_{l_N k_N}\dots g_{l_{p+1}k_{p+1}}\xi^{l_{p+1}}\dots\xi^{l_N}\varepsilon^{k_N\dots k_{p+1}i_1\dots i_p}\\
&= c_p q^{-N(p-N/2)}\Lambda^{2p-N}\xi^{l_{p+1}}\dots\xi^{l_N}\varepsilon_{l_N\dots l_{p+1}}{}^{i_1\dots i_p}\\
&= \text{rhs}(5.6)
\end{aligned}
$$

Proof of proposition 9

We now evaluate the lhs (5.10) on each $\mathcal{DC}^p$. We find

$$
\begin{aligned}
(\mathrm{d}^*\mathrm{d}^* + {}^*\mathrm{d}^*\mathrm{d})\mathbf{1} &= {}^*\mathrm{d}^*\mathrm{d}\mathbf{1} = {}^*\mathrm{d}^*\xi^{i_1}\partial_{i_1}\\
&= q^{-N(1-\frac{N}{2})}c_1\,{}^*\mathrm{d}\xi^{i_2}\dots\xi^{i_N}\varepsilon_{i_N\dots i_2}{}^{i_1}\Lambda^{2-N}\partial_{i_1}\\
&= (-1)^{N-1}q^{-N(1-\frac{N}{2})}c_1\,{}^*\xi^{i_2}\dots\xi^{i_N}\,\mathrm{d}\varepsilon_{i_N\dots i_2}{}^{i_1}\Lambda^{2-N}\partial_{i_1}\\
&= (-1)^{N-1}q^{-N}c_1c_N\varepsilon^{i_2\dots i_N j}\Lambda^N\partial_j\varepsilon_{i_N\dots i_2}{}^{i_1}\Lambda^{2-N}\partial_{i_1}\\
&= q^2c_1c_N U^{-1}{}_h^j\varepsilon^{hi_2\dots i_N}\partial_j\varepsilon_{i_N\dots i_2}{}^{i_1}\partial_{i_1}\Lambda^2\\
&= q^2c_1c_N g^{lj}\varepsilon_l{}^{i_2\dots i_N}\partial_j\varepsilon_{i_N\dots i_2}{}^{i_1}\partial_{i_1}\Lambda^2\\
&= \frac{q^2c_1c_N}{d_1}g^{i_1 j}\partial_j\partial_{i_1}\Lambda^2 = \frac{q^2c_1c_N}{d_1}\,\Box\Lambda^2\\
&= \frac{q^2c_1}{c_0[N]_q}\,\frac{q^{-\frac{N}{2}}+q^{\frac{N}{2}}}{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}\,\Box\Lambda^2
\end{aligned}
$$

for $p=0$, for $p=N$

$$
\begin{aligned}
(\mathrm{d}^*\mathrm{d}^* + {}^*\mathrm{d}^*\mathrm{d})\mathrm{d}^N x &= \mathrm{d}^*\mathrm{d}^*\mathrm{d}^N x = q^{-\frac{N^2}{2}}c_N\,\mathrm{d}^*\mathrm{d}\Lambda^N = q^{-\frac{N^2}{2}}c_N\,\mathrm{d}^*\xi^{i_1}\partial_{i_1}\Lambda^N\\
&= q^{-N}c_Nc_1\,\mathrm{d}\xi^{i_2}\dots\xi^{i_N}\varepsilon_{i_N\dots i_2}{}^{i_1}\Lambda^{2-N}\partial_{i_1}\Lambda^N\\
&= (-1)^{N-1}q^{2-2N}c_Nc_1\xi^{i_2}\dots\xi^{i_N}\,\mathrm{d}\varepsilon_{i_N\dots i_2}{}^{i_1}\partial_{i_1}\Lambda^2\\
&= (-1)^{N-1}q^{2-2N}c_Nc_1\xi^{i_2}\dots\xi^{i_N}\xi^j\partial_j\varepsilon_{i_N\dots i_2}{}^{i_1}\partial_{i_1}\Lambda^2\\
&= (-1)^{N-1}q^{2-2N}c_Nc_1\varepsilon^{i_2\dots i_N j}\,\mathrm{d}^N x\,\varepsilon_{i_N\dots i_2}{}^{i_1}\partial_j\partial_{i_1}\Lambda^2
\end{aligned}
$$

$$
\begin{aligned}
&= q^2 c_N c_1 g^{lj} \varepsilon_l{}^{i_2 \dots i_N} \varepsilon_{i_N \dots i_2}{}^{i_1} \partial_j \partial_{i_1} \Lambda^2 \, \mathrm{d}^N x \\
&= \frac{q^2 c_1 c_N}{d_1} g^{i_1 j} \partial_j \partial_{i_1} \Lambda^2 \, \mathrm{d}^N x = \frac{q^2 c_1 c_N}{d_1} \Box \Lambda^2 \, \mathrm{d}^N x \\
&= \frac{q^2 c_1}{c_0 [N]_q} \frac{q^{-\frac{N}{2}} + q^{\frac{N}{2}}}{q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1}} \Box \Lambda^2 \, \mathrm{d}^N x
\end{aligned}
$$

for $p = N$, whereas for $p = 1, 2, \dots, N-1$ we find on one hand

$$
\begin{aligned}
\mathrm{d}^* \mathrm{d}^* \xi^{i_1} \dots \xi^{i_p} &\overset{(5.6)}{=} (-1)^{N-p} q^{-N(p-\frac{N}{2})+N-2p} c_p \, \mathrm{d}^* \xi^{i_{p+1}} \dots \xi^{i_N} \varepsilon_{i_N \dots i_{p+1}}{}^{i_1 \dots i_p} \Lambda^{2p-N} \, \mathrm{d} \\
&\overset{(5.6)}{=} (-1)^{N-p} q^{-N(p-\frac{N}{2})+N-2p-N(N-p+1-\frac{N}{2})} c_p c_{N-p+1} \, \mathrm{d} \\
&\quad \times \xi^{h_1} \dots \xi^{h_{p-1}} \varepsilon_{h_{p-1} \dots h_1}{}^{i_{p+1} \dots i_N i} \varepsilon_{i_N \dots i_{p+1}}{}^{i_1 \dots i_p} \Lambda^2 \partial_i \\
&\overset{(3.12)}{=} (-1)^{N-1} q^{-2p} c_p c_{N-p+1} \xi^{h_1} \dots \xi^{h_{p-1}} \, \mathrm{d} (-1)^{N-1} \\
&\quad \times g^{l_p i} \varepsilon_{l_p h_{p-1} \dots h_1}{}^{i_{p+1} \dots i_N} \varepsilon_{i_N \dots i_{p+1}}{}^{i_1 \dots i_p} \Lambda^2 \partial_i \\
&\overset{(3.18)}{=} q^{2-2p} \frac{c_p c_{N-p+1}}{d_p} \xi^{h_1} \dots \xi^{h_{p-1}} \, \mathrm{d} \mathcal{P}^{a,p}{}_{h_1 \dots h_{p-1} j_p}^{i_1 \dots i_{p-1} i_p} \partial^{j_p} \Lambda^2
\end{aligned}
$$

and on the other

$$
\begin{aligned}
{}^*\mathrm{d}^* \, \mathrm{d} \xi^{i_1} \dots \xi^{i_p} &= (-1)^p {}^*\mathrm{d}^* \xi^{i_1} \dots \xi^{i_p} \mathrm{d} \\
&\overset{(5.6)}{=} (-1)^p c_{p+1} q^{-N(p+1-\frac{N}{2})*} \, \mathrm{d} \xi^{i_{p+2}} \dots \xi^{i_N} \varepsilon_{i_N \dots i_{p+2}}{}^{i_1 \dots i_{p+1}} \Lambda^{2p+2-N} \partial_{i_{p+1}} \\
&= (-1)^{N-1} c_{p+1} q^{-N(p-\frac{N}{2})-2p-2*} \xi^{i_{p+2}} \dots \xi^{i_N} \xi^{j_N} \varepsilon_{i_N \dots i_{p+2}}{}^{i_1 \dots i_{p+1}} \Lambda^{2p+2-N} \partial_{j_N} \partial_{i_{p+1}} \\
&\overset{(5.6)}{=} (-1)^{N-1} c_{p+1} c_{N-p} q^{-2p-2} \xi^{h_1} \dots \xi^{h_p} \varepsilon_{h_p \dots h_1}{}^{i_{p+2} \dots i_N j_N} \varepsilon_{i_N \dots i_{p+2}}{}^{i_1 \dots i_{p+1}} \Lambda^2 \partial_{j_N} \partial_{i_{p+1}} \\
&\overset{(3.12)}{=} c_{p+1} c_{N-p} q^{-2p-2} \xi^{h_1} \dots \xi^{h_p} g^{h_{p+1} j_N} \varepsilon_{h_{p+1} h_p \dots h_1}{}^{i_{p+2} \dots i_N} \varepsilon_{i_N \dots i_{p+2}}{}^{i_1 \dots i_{p+1}} \Lambda^2 \partial_{j_N} \partial_{i_{p+1}} \\
&\overset{(3.18)}{=} q^{2-2p} \frac{c_{p+1} c_{N-p}}{d_{p+1}} \xi^{h_1} \dots \xi^{h_p} \mathcal{P}^{a,p+1}{}_{h_1 \dots h_{p+1}}^{i_1 \dots i_{p+1}} g^{h_{p+1} j_N} \partial_{j_N} \partial_{i_{p+1}} \Lambda^2 \\
&\overset{(2.4)}{=} \frac{q^{2-2p}}{[p+1]_q} \frac{c_{p+1} c_{N-p}}{d_{p+1}} \xi^{h_1} \dots \xi^{h_p} \mathcal{P}^{a,p}{}_{h_1 \dots h_{p-1} j_p}^{i_1 \dots i_{p-1} i_p} \\
&\quad \times \left[q^p \, \delta^{j_p}_{h_p} g^{i_{p+1} j_N} - [p]_q \hat{R}^{j_p i_{p+1}}_{h_p h_{p+1}} g^{h_{p+1} j_N} - \frac{k [p]_q}{1 + q^{N-2p}} \delta^{j_N}_{h_p} g^{j_p i_{p+1}} \right] \partial_{j_N} \partial_{i_{p+1}} \Lambda^2 \\
&\overset{(1.7)}{=} \frac{q^{2-2p}}{[p+1]_q} \frac{c_{p+1} c_{N-p}}{d_{p+1}} \xi^{h_1} \dots \xi^{h_p} \mathcal{P}^{a,p}{}_{h_1 \dots h_{p-1} j_p}^{i_1 \dots i_{p-1} i_p} \\
&\quad \times \left[q^p \delta^{j_p}_{h_p} g^{i_{p+1} j_N} - [p]_q g^{j_p h_{p+1}} \hat{R}^{-1}{}^{i_{p+1} j_N}_{h_{p+1} h_p} - \frac{k [p]_q}{1 + q^{N-2p}} \delta^{j_N}_{h_p} g^{j_p i_{p+1}} \right] \partial_{j_N} \partial_{i_{p+1}} \Lambda^2 \\
&\overset{(1.13),(1.1)}{=} \frac{q^{2-2p}}{[p+1]_q} \frac{c_{p+1} c_{N-p}}{d_{p+1}} \xi^{h_1} \dots \xi^{h_p} \mathcal{P}^{a,p}{}_{h_1 \dots h_{p-1} j_p}^{i_1 \dots i_{p-1} i_p} \left[\delta^{j_p}_{h_p} g^{i_{p+1} j_N} \left(q^p - [p]_q \frac{k}{\mu} \right) \right. \\
&\quad \left. - [p]_q \left(q^{-1} + \frac{k}{1 + q^{N-2p}} \right) \delta^{j_N}_{h_p} g^{j_p i_{p+1}} \right] \partial_{j_N} \partial_{i_{p+1}} \Lambda^2 \\
&= \frac{q^{2-2p}}{[p+1]_q} \frac{c_{p+1} c_{N-p}}{d_{p+1}} \left[\frac{q^{p+1-\frac{N}{2}} + q^{\frac{N}{2}-p-1}}{q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1}} \xi^{i_1} \dots \xi^{i_p} \Box \right. \\
&\quad \left. - [p]_q \frac{q^{p+1-\frac{N}{2}} + q^{\frac{N}{2}-p-1}}{q^{p-\frac{N}{2}} + q^{\frac{N}{2}-p}} \xi^{h_1} \dots \xi^{h_{p-1}} \, \mathrm{d} \mathcal{P}^{a,p}{}_{h_1 \dots h_{p-1} j_p}^{i_1 \dots i_{p-1} i_p} \partial^{j_p} \right] \Lambda^2 .
\end{aligned} \tag{A.20}
$$

In order that the second term in the square bracket be opposite of $\mathrm{d}^*\mathrm{d}^*\xi^{i_1}\dots\xi^{i_p}$ it must be

$$0 \stackrel{!}{=} \frac{c_p c_{N-p+1}}{d_p} - \frac{q^{p+1-\frac{N}{2}}+q^{\frac{N}{2}-p-1}}{q^{p-\frac{N}{2}}+q^{\frac{N}{2}-p}}\frac{[p]_q}{[p+1]_q}\frac{c_{p+1}c_{N-p}}{d_{p+1}}$$
$$= \frac{q^{-\frac{N}{2}}+q^{\frac{N}{2}}}{q^{p-\frac{N}{2}}+q^{\frac{N}{2}-p}}\frac{[p]_q![N-p-1]_q!}{[N]_q!d_N}\{[N-p]_q c_p c_{N-p+1} - [p]_q c_{p+1}c_{N-p}\}$$

namely, for $p = 1, 2, \dots, N-1$

$$[N-p]_q c_p c_{N-p+1} - [p]_q c_{p+1}c_{N-p} = 0.$$

This recursion relation is solved by

$$c_{p+1}c_{N-p} = \binom{[N-1]_q}{[p]_q} c_1 c_N. \tag{A.21}$$

When replaced in (A.20) we find, on all of $\mathcal{DC}^*$, and in particular on Ω^*,

$$(\mathrm{d}^*\mathrm{d}^* + {}^*\mathrm{d}^*\mathrm{d})\xi^{i_1}\dots\xi^{i_p} = \frac{q^2 c_1}{c_0[N]_q}\frac{q^{-\frac{N}{2}}+q^{\frac{N}{2}}}{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}\,\Box\Lambda^2\xi^{i_1}\dots\xi^{i_p}. \tag{A.22}$$

Taking $\prod_{p=1}^{N-2}$ of both sides of (A.21) and multiplying the result by $c_0(c_1c_N)^2$ we obtain

$$d_1 d_2 \dots d_N c_N = c_0(c_1c_N)^N \binom{[N-1]_q}{[1]_q}\dots\binom{[N-1]_q}{[N-2]_q},$$

implying, because of (3.17),

$$\left(\frac{c_1}{c_0[N]_q}\right)^N = \left(\frac{c_1c_N}{d_N[N]_q)}\right)^N = \frac{c_N}{c_0[N]_q!}\prod_{l=0}^{N-1}\frac{q^{l-\frac{N}{2}}+q^{\frac{N}{2}-l}}{q^{-\frac{N}{2}}+q^{\frac{N}{2}}}.$$

Before proceeding we note that we are still free to choose the value of $d_N = c_0c_N$ and the normalization of c_N w.r.t. c_0, in other words we are free to choose the values of both c_0, c_N. We choose

$$c_N = 1, \qquad c_0 = d_N = \frac{1}{[N]_q!}\prod_{l=0}^{N-1}\frac{q^{l-\frac{N}{2}}+q^{\frac{N}{2}-l}}{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}; \tag{A.23}$$

the first choice guarantees that ${}^*\mathbf{1} = \mathrm{d}V$, and therefore also ${}^*\mathrm{d}V = \mathbf{1}$ in view of $*^2$. As a consequence,

$$\left(\frac{c_1}{c_0[N]_q}\right)^N = \left(\frac{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}{q^{-\frac{N}{2}}+q^{\frac{N}{2}}}\right)^N$$

implying

$$c_1 = c_0[N]_q\frac{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}{q^{-\frac{N}{2}}+q^{\frac{N}{2}}} = \frac{1}{[N-1]_q!}\prod_{l=1}^{N-1}\frac{q^{l-\frac{N}{2}}+q^{\frac{N}{2}-l}}{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}.$$

Multiplying both sides of (A.21) by $1/c_{N-p} = c_p/d_p$ and using (3.17) we thus obtain the recursion relation

$$c_{p+1} = c_p[N-p]_q\frac{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}{q^{p-\frac{N}{2}}+q^{\frac{N}{2}-p}}.$$

Solving the latter, the final result for $p = 0, 1, \dots, N$ reads (5.11). As for (A.22), we find that on all of $\mathcal{DC}^*$, and in particular on Ω^*, (5.10) holds.

Finally, we see that the normalization condition (A.23) for $d_N = d_0$ implies a specific value for the normalization constant

$$\gamma_N \equiv \begin{cases} (\varepsilon^{12\ldots N})^{-1} \\ (\varepsilon^{-n(1-n)\ldots n})^{-1} \end{cases}$$

in (3.5), which can be computed case by case. In particular, for the cases $N = 3, 4$ this implies a different normalization w.r.t. the one adopted in section 3.

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